

ORE'S THEOREM, LABELLED GRAPHS, FACEBOOK

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ABSTRACT

In continuation of our work on the application of graph theory and in particular labeled graph theory to SNS, we examine in this paper the applications of Ore's theorem to SNS. Ore's theorem gives a sufficient condition for the existence of a closed Hamiltonian Path and application of Ore's theorem to Facebook requires some new definitions of two labeled degree of a vertex and labeled Hamiltonian paths. In this paper, each definition is examined for its application to SNS. Under these definitions, Ore's theorem and its converse is verified in Facebook. Though in general converse of Ore's theorem is not valid, we demonstrate how it works in SNS with some altered conditions. We also show that a closed Hamiltonian path in Facebook exists for any two nonadjacent vertices without depending on the condition of Ore's theorem. Hamiltonian paths by definition result in a one way communication paths in Facebook between two non adjacent vertices, this becomes a two way communication path, thanks to Ore's theorem which guarantees a closed Hamiltonian path.

KEYWORDS: Directed Hamiltonian Path, Hamiltonian Path, Hamiltonian Cycle, Hamiltonian Graph Labeled Degree of a Vertex, Path

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INTRODUCTION

Graph theory became popular and mathematicians, scientists, engineers, sociologists, economists etc., looked towards graph theory for applications in their fields. In keeping this trend, we apply Ore's Theorem to Labeled graphs and then apply it to Facebook. As Facebook /network is the most popular area which enables many people to interact, we want to investigate the mathematical background behind this new area by application of graph theory to Facebook.

In our earlier work, we applied various concepts, theorems and results in graph theory to Facebook. This paper is devoted to application of a classical theorem of Ore's to Facebook. In this connection, we formulate some essential definitions by altering the usual standard definitions as well as translating some theorems so as to apply these to Facebook.

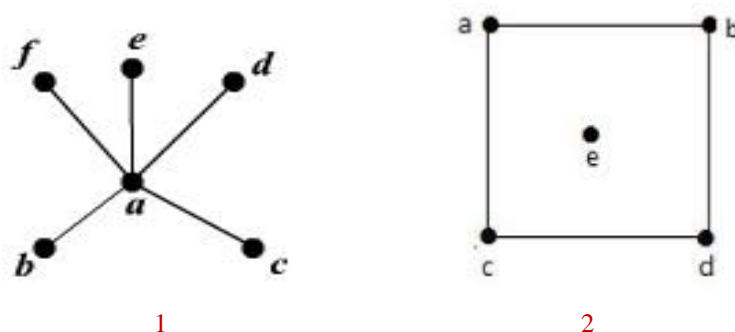
MATHEMATICAL PRELIMINARIES

In this section, we define certain concepts in graph theory. Concepts not defined here are understood as they are in any standard book on Graph theory (reference:[3]). We claim no originality in this section save perhaps for its presentation.

For definitions of Graph, label, Labeled Vertex and Labeled Edge, Labeled Graphs, Path, Trail, labeled path, Menger's theorem, semi labeled graphs, mixed labeled graphs and other definitions refer papers in reference: [5] and [6]

Degree of a Vertex

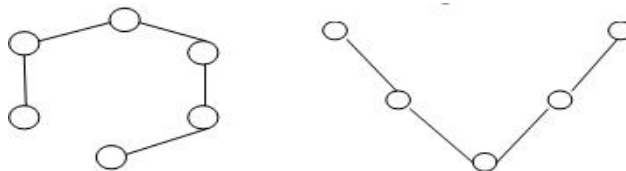
In graph theory, the degree of a vertex is the number of edges connecting it. In the example below, vertex a has degree 5, and the rest have degree 1. A vertex with degree 1 is called an "end vertex". In figure (2), vertices a,b,c,d have degree 2 and vertex e is an isolated point.



Hamiltonian Path

A Hamiltonian path is a path that visits each vertex of the graph exactly once. A graph that contains a Hamiltonian path is called a Traceable Graph. Also a Hamiltonian cycle is a Hamiltonian path that is a cycle.

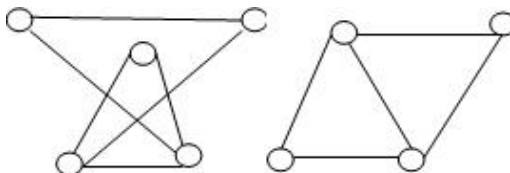
Example



Hamiltonian Graph

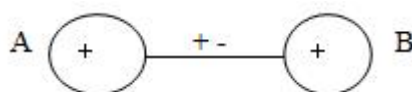
A graph is Hamiltonian if for every pair of vertices there is a Hamiltonian path between the two vertices.

Example



First Label of a 2-Labeled Edge

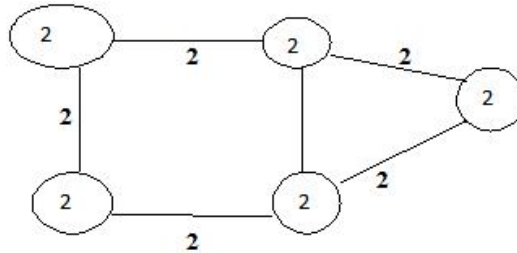
If AB is an edge with a label '+ -' then the first label is +, in this case the first label of BA is -.



Hamiltonian Labeled Path

A Hamiltonian labeled path in a graph is a Hamiltonian path all of whose vertices and edges have the same label. A graph that contains a Hamiltonian labeled path is called a Traceable labeled Graph. Also a Hamiltonian labeled cycle is a Hamiltonian labeled path that is a cycle.

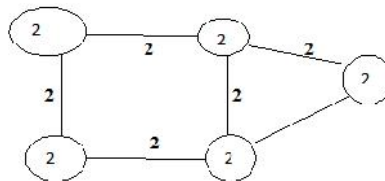
Example



Hamiltonian Labeled Graph

A graph is Hamiltonian labeled if for every pair of labeled vertices there is a Hamiltonian labeled path between the two labeled vertices.

Example

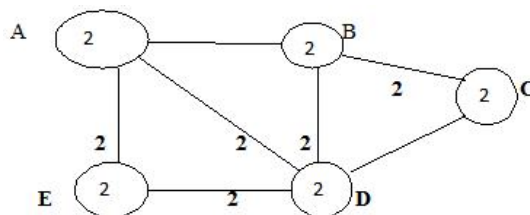


Difference between Classical Hamiltonian Path and Hamiltonian Labeled Path

Example

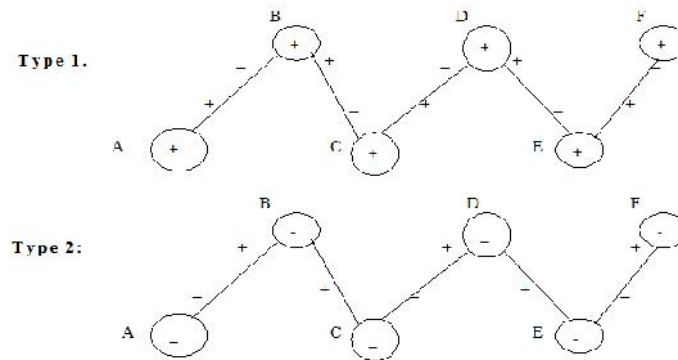
In the graph below Hamiltonian Path is ABCDE

But Hamiltonian Labeled Path is AEDBC



Hamiltonian Labeled Path in a 2-Labeled Edge Graph

A Hamiltonian labeled path in a 2-labeled edge graph whose vertices are single labeled is a Hamiltonian path where the label of the vertex and the 1st label of the edge are same.



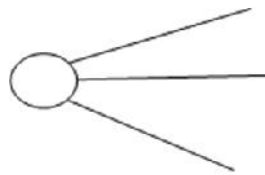
However in Facebook the 2nd type of Hamiltonian path is not defined.

Labeled Degree of a Vertex: ($deg_l v$):

In the usual definition of the degree of the vertex in an unlabeled graph depends on the number of edges incident on the particular vertex but for a labeled graph we have to define degree in different ways.

1. Let v be any vertex (unlabeled) then the usual degree of v *is equal to* the number of unlabeled edges incident on v .
i.e., $deg(v)=3$

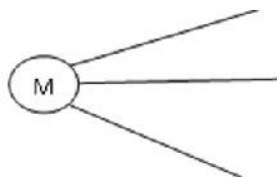
Example



In this example the degree is 3

2. Let v be a vertex with label M then if 'n' unlabeled edges are incident on v then degree of the vertex v is equal to 0. i.e., $deg(v)=0$

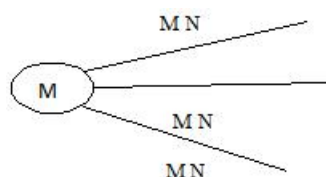
Example



In this example the degree is 0

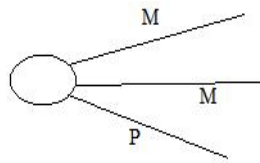
3. Let v be a vertex with label M then if 'n' 2-labeled edges with one of the label M are incident on v then degree of v is equal to 'n'. i.e., $deg(v) = n$. This is same as definition 1.

Example



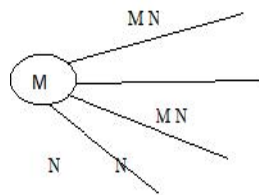
In this example the degree is 3

4. Let v be a unlabeled vertex then if 'n' labeled edges are incident on v then degree of vertex v is equal to 0. i.e., $deg(v)=0$. This is same as definition 2.

Example

In this example the labeled degree is 0 and usual degree is 3

5. If 'm' edges with label 'M' are incident on a vertex v of label 'M' and 'n' edges with label 'N' are incident on same vertex v then degree of the vertex v is equal to 'm'.

Example

In this example the labeled degree is 2 and usual degree is 4

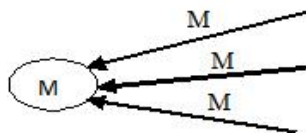
Note 1: Degree of a vertex will always be greater than or equal to than the labeled degree of a vertex.

n-Labeled Degree

A n-labeled degree is the number of same labeled n edges incident on a vertex v of the same label.

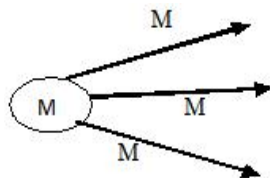
Labeled Degree of a vertex in Directed Graph: ($dd_{\ell} v$):

1) **Labeled In-Degree:** Let v be a vertex with label M then if 'n' directed edges with the same label M are incident (incoming) on labeled vertex v then degree of v is equal to 'n'. i.e., $\deg(v)=n$.

Example

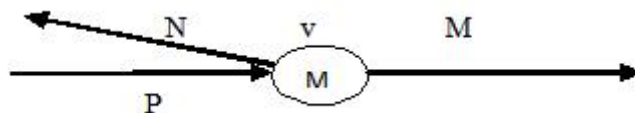
In this example the degree is 3

2) **Labeled Out-degree:** Let v be a vertex with label M then if 'n' directed edges with the same label M are outgoing from labeled vertex v then degree of v is equal to 'n'. i.e., $\deg(v)=n$.

Example

In this example the degree is 3

Note 2: in the figure below the labeled out-degree (v)=1 and labeled in-degree(v)=0

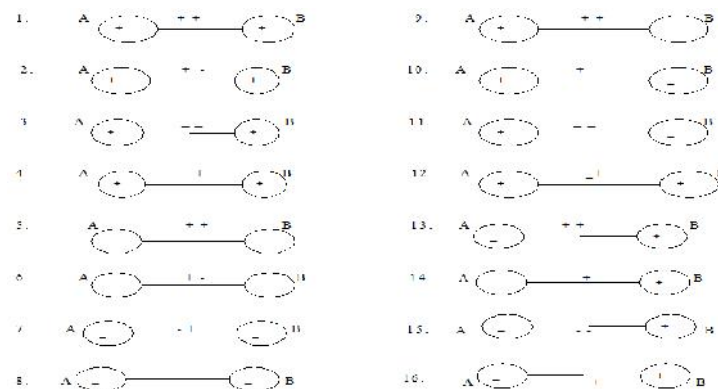


Note 3

Degree of a vertex in a simple graph will always be greater than the labeled degree of a vertex in a directed graph.

Note 4

If an edge AB has 2 labels and vertices A and B are single labeled then there are 16 types of edges between A and B which are as follows:



In all the above relations, AB is an edge but in Facebook, there is no edge when one of them is offline. In general the above relations make A and B adjacent with labeled edges. However we define two vertices to be non adjacent if the label of the edge AB is '- -' (this is applied in Facebook), whereas the remaining relations do not have physical meaning when applied to Facebook. We can also define A and B as non adjacent if the label of the edge is '+ -' or '- +' but in this case the proof of the theorem is a bit involved and will be taken up elsewhere. However we can have a Hamiltonian path under this definition also.

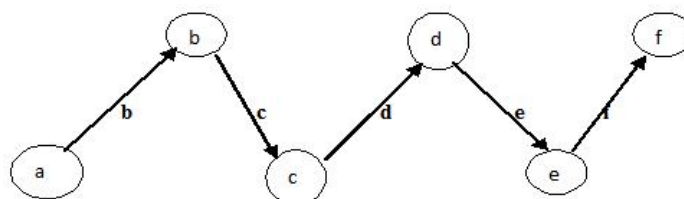
In Facebook there is no relation between 2 people when one of them is offline.

Adjacent Vertices

We know that if AB is an edge then A and B are adjacent. In the case of Labeled graphs where both edges and vertices are labeled the definition of adjacent vertices is not unique, each definition has its own application in Facebook. We define adjacent vertices corresponding to each of the above relations.

Hamiltonian Two Labeled Path**1) Directed Hamiltonian Labeled Path**

A labeled path a, b, c, d, e, f is called a directed Hamiltonian labeled path in a directed graph if the sequence of edges and vertices between a and f have the same label.

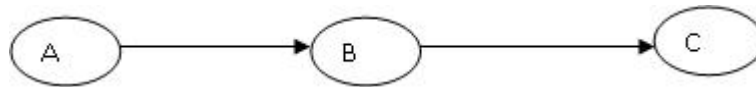
Example

Directed Hamiltonian labeled path from a to f is $abcdef$

Directed Hamiltonian Labeled Graph

A graph is Directed Hamiltonian labeled if for every pair of labeled vertices there is a directed Hamiltonian labeled path between them.

Note 5: Directed path is a path in which there is a directed edge between two vertices.



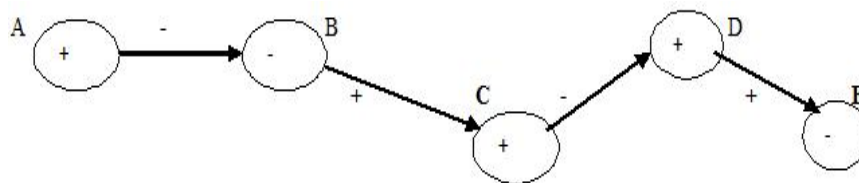
Directed path is ABC but CBA is not a directed path.

Directed Hamiltonian Mixed Labeled Path

A labeled path a, b, c, d, e is called a directed Hamiltonian mixed labeled path in a directed graph if the directed path between a and e is a mixed labeled path. reference [2].

Example

Directed Hamiltonian mixed labeled path from a to e is $abcde$



Directed Hamiltonian Mixed Labeled Graph

A graph is Directed Hamiltonian mixed labeled if for every pair of labeled vertices there is a Directed Hamiltonian mixed labeled path between the two labeled vertices.

Basic Definitions in Terms of Facebook

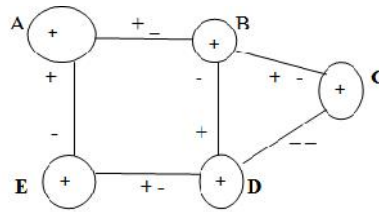
Definitions like Vertex, Labeled Vertex, Edge, Edge for Multi Labeled vertices, Labeled Edge, Single Labeled Edge, Multi Labeled Edges, refer the papers in reference [5] and [6]

Note 6: There are 16 ways (as defined earlier under note 4:) to define adjacency of two vertices in Facebook but we consider only for online people.(vertices with label '+').

Hamiltonian Labeled path in Facebook

A Hamiltonian labeled path in a graph of a particular group of a Facebook network is a labeled path, which visits all the people who are online such that there is a one way communication (OR) one way transfer of information from one person to another.

In this case, since it is one way transfer of information, sharing of information is defined with a label '+' on the respective edge between two vertices each with label '+'. For example



In the above graph of a particular group of a Facebook network, the Hamiltonian labeled path is AEDBC.

Note 7

In a closed Hamiltonian path every person gets the information sent by anybody in the given situation of a particular group of a Facebook network.

Hamiltonian Labeled Graph of Facebook

A graph is Hamiltonian labeled in Facebook if for every pair of online people there is a Hamiltonian labeled path of Facebook between the two online people.

Adjacency in Facebook

Let us consider the 16 relations as mentioned earlier under Note 2, if we consider any relation among the 16 relations then we get adjacent vertices in general. In Facebook we define adjacency only for online people. *In Facebook there is no relation between 2 people when one of them is offline.* Two online people are non adjacent when there is no flow of information between them(In both the ways).

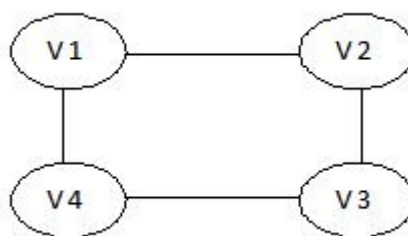
When visiting a country, one likes to visit every place only once by taking the shortest route, the shortest closed Hamiltonian path is very much suitable in this case. Before we find such a path we should know when such a path can exist Ore's theorem answers this question.

ORE'S THEOREM: (Statement)

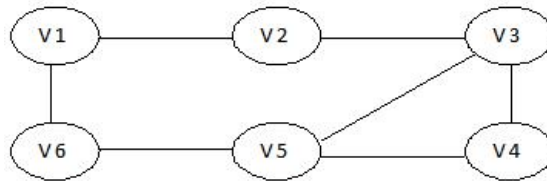
Let G be a simple graph on n vertices. If $n \geq 3$, and $\text{degree}(x) + \text{degree}(y) \geq n$ for each pair of non-adjacent vertices x and y , then G has a closed Hamiltonian path.

Let us consider *an example*

Example



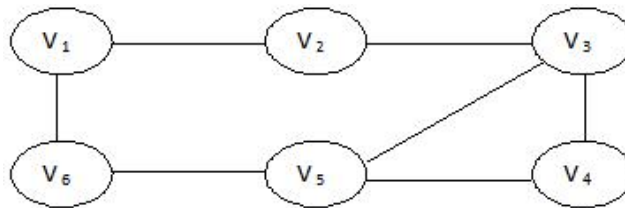
In this eg., Consider the two non adjacent vertices v_1 and v_3 $\text{deg } v_1=2$ and $\text{deg } v_3=2$ Therefore $\text{deg } v_1 + \text{deg } v_3$ number of vertices $\Rightarrow 2+2=4 \Rightarrow$ the above graph has a closed Hamiltonian path

Example

In the eg above., Consider the two non adjacent vertices v_1 and v_3 $\deg v_1=2$ and $\deg v_3=3$. Therefore $\deg v_1 + \deg v_3 \Rightarrow 2 + 3 = 5 \not\geq 6$ where $n=6$ is the number of vertices but the above graph has a closed Hamiltonian path. The above examples show that the **condition in the ore's theorem is not necessary for the existence of a closed Hamiltonian path.**

Converse of Ore's Theorem

Let G be a simple graph on ' n ' vertices, if $n \geq 3$ and also for each pair of non-adjacent vertices x and y G has a closed Hamiltonian path then " $\deg(x) + \deg(y) \geq n$ ".

Example

In this eg., The Closed Hamiltonian path is $v_1, v_2, v_3, v_4, v_5, v_6, v_1$ Consider the two non adjacent vertices v_1 and v_3 $\deg v_1=2$ and $\deg v_3=3$ Therefore $\deg v_1 + \deg v_3 \not\geq$ number of vertices $\Rightarrow 2+3=5 \not\geq 6$ The above graph has a closed Hamiltonian path even though $\deg(x) + \deg(y) \not\geq n$. Therefore converse is not true.

Note 8

Consider the graph above, if $\deg(x) + \deg(y) \geq 6$ (say) then this condition has the following possibilities:

- If one of them is zero then the condition is not satisfied in this there is no closed Hamiltonian path.
- If $\deg(x)=1$ and $\deg(y)=1$ then also the condition is not satisfied.
- If $\deg(x)=2$ and $\deg(y)=2$ then also the condition is not satisfied, and so on

The condition here denotes that online person cannot converse to all at a given instant of time.

Note 9

- To translate Ore's theorem to labeled graphs we take into account the definitions of various concepts in labeled graphs as given in mathematical preliminaries.
- We can state Ore's theorem under several definitions of labeled degree of a vertex defined in the mathematical preliminaries but we prove the theorem for definition 3 of labeled degree of a labeled vertex.

ORE'S THEOREM UNDER DEFINITION 3 OF LABELED DEGREE OF A VERTEX

Statement: Let G be a labeled graph on n ($n \geq 3$) vertices with exactly one pair of non adjacent vertices (according to relation 3) x and y such that $\deg_l(x) + \deg_l(y) \geq n$ for each pair of non-adjacent labeled vertices x and y , then G has a closed Hamiltonian labeled path or a Hamiltonian Graph.

Proof: Let G be labeled graph on ' n ' vertices where $n \geq 3$, we assume that G satisfies the condition $\deg_l(x) + \deg_l(y) \geq n$ and is not a Hamiltonian graph.

Here G has a pair of non-adjacent labeled vertices (according to relation 3). We add labeled edge (according to definition 3 of labeled degree of a vertex) to the graph by joining all the non adjacent labeled vertices until we obtain a graph H such that the addition of one more labeled edge joining non adjacent labeled vertices in H will produce a Hamiltonian graph with ' n ' labeled vertices.

Let x and y be two labeled vertices which are not adjacent in H , they are non adjacent in G also. (This is the only pair of non adjacent vertices in G).

G satisfies the condition i.e., $\deg_l(x) + \deg_l(y) \geq n$ (given). This condition is satisfied in H also.

If we add a labeled edge (3rd relation) between the non adjacent vertices x and y then the graph is a Hamiltonian graph. In graph H there will be a Hamiltonian Labeled path between x and y

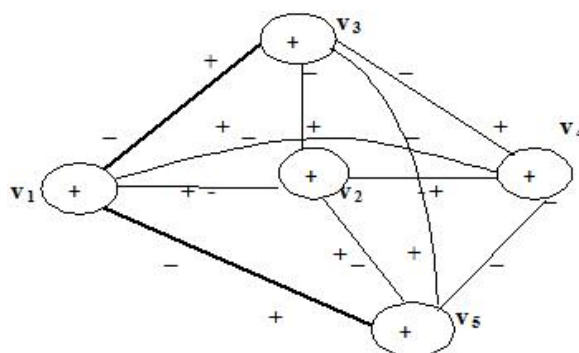
If $x=v_1$ and $y=v_n$ then the resulting Hamiltonian labeled path is $v_1, v_2, \dots, v_{i-1}, v_i, v_{i+1}, \dots, v_n$, let labeled degree of v_1 be r . If v_1 and v_i are adjacent with labeled edge (according to definition 3 of labeled degree of a vertex) and also v_{i-1} and v_n are adjacent with labeled edge (according to definition 3 of labeled degree of a vertex) then H is a Hamiltonian graph.

Therefore only v_1 and v_i should be adjacent with a labeled edge, and v_{i-1} and v_n should not be adjacent, this is true for $n=2, 3, \dots, (n-1)$. Therefore $\deg_l(v_n) \leq (n-1) - r \Rightarrow \deg_l(v_n) + \deg_l(v_1) \leq n-1$, which is the contradiction to our assumption.

Therefore G has a closed Hamiltonian labeled path.

Verification of Ore's Theorem under the Definition 3 of Labeled Degree of a Vertex

Example



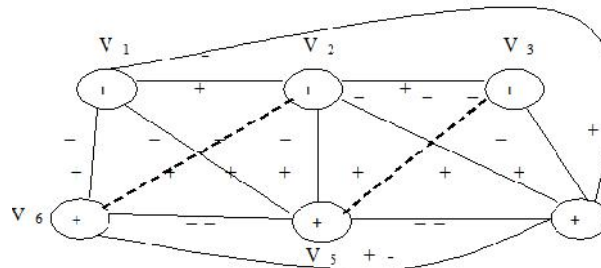
According to the definition of non adjacency (from the relation 3), v_4 and v_5 are non adjacent. $\deg_l(v_4) + \deg_l(v_5) = 3 + 3 = 6 \geq 5$, here 5 is the number of vertices (n).

Therefore G has a closed Hamiltonian Path: $\{v_5, v_3, v_1, v_4, v_2, v_5\}$.

Note 10

If we have more than one pair of non adjacent labeled vertices then the theorem may not be true as shown in the example below:

Example



In the above graph there are two pairs of non adjacent vertices (according to 3-relation) i.e., V_5 and V_4 , V_5 and V_6

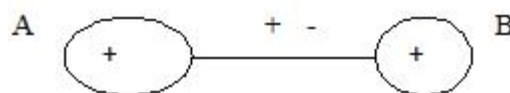
Here $\deg_l V_4 + \deg_l V_5 = 3 + 3 = 6 = n$ ($n=6$)

$\deg_l V_5 + \deg_l V_6 = 3 + 3 = 6 = n$ ($n=6$)

But we see that there is no Hamiltonian path in the graph. As a result, we consider Ore's theorem in Facebook with a condition that there should be only one pair of non-adjacent vertices. Whether the Facebook admits no Hamiltonian path if the Facebook has more than one pair of non-adjacent online persons is not taken up here.

Labeled Degree of a Vertex in Facebook

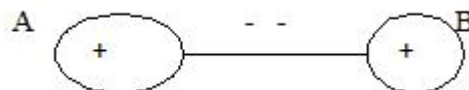
In Facebook we take the degree of A as 1 if AB has a label '+' i.e.,



In the above figure degree of $A=1$ and degree of $B=0$

Note 11

If at one time there are only 2 online people who do not share any information between them then we see that under relation 3 Ore's theorem's condition satisfies but there is no closed Hamiltonian path. i.e., $\deg(A) + \deg(B) \geq 2$



We consider only groups of a particular Facebook network in order to apply Ore's theorem.

Ore's Theorem in Terms of Facebook: (Labeled Edges)

Statement: Let G be a particular group of a Facebook network with number of people online(n) ≥ 3 ,

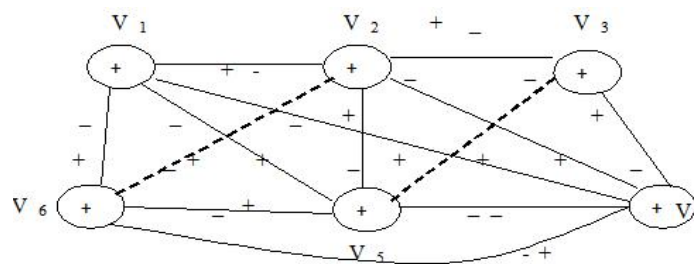
we denote $\deg_1 x$ as the number of people with whom x converses or shares information (here label edge '+' indicates the flow of information and '-' indicates there is no flow of information).

Then the Ore's theorem in terms of Facebook states that if the of people with whom x converses and the number of people with whom y converses \geq the number of people in a particular group of a face network at a given time where the **group has exactly one pair of non adjacent online vertices.**, i.e., $\deg_1 x + \deg_1 y \geq n$ for every pair of people who are not conversing directly (i.e., 3rd relation), then the network is **Hamiltonian labeled graph** of Facebook.

Proof: Here the proof is on the same lines as on labeled Ore's theorem given above (according to definition 3 of labeled degree of the labeled vertex). **Also, we note here that if there are exactly 2 online people in the particular group of Facebook network who are sharing information or conversing then this theorem is not applicable.**

Verification of Ore's Theorem in Facebook

Example



According to the definition of adjacency in Facebook network, in the graph above we see that there is no one way transfer of information between one pair of vertices i.e., v_4 and v_5

Here $\deg_1 V_4 + \deg_1 V_5$

$$= 3 + 3 = 6 = n \text{ where } n=6 \text{ (number of people in the network)}$$

Therefore there G has a closed Hamiltonian path: $\{V_4, V_6, V_1, V_2, V_5, V_3, \text{ and } V_4\}$

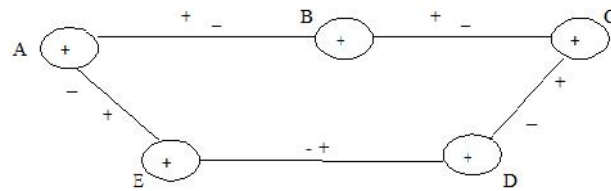
Therefore Ore's theorem is satisfied for a Facebook group's network.

Facebook Ore's Theorem has to get the additional condition that neither $\deg_1 x$ nor $\deg_1 y = 0$. In this case Ore's theorem ensures that there is some person conversing with both x and y .

i.e., If there is a person conversing with both x and y then there is a closed Hamiltonian path.

In a one way conversation any online person can send information to any other online person if there is an Hamiltonian Path.

Therefore Ore's theorem when applied to Facebook guarantees a 2-way communication/conversation as shown in the example below:



The information is traveled from A, B, C, D, E, and again to A.

Verification of Converse of Ore's Theorem in Terms of Facebook

Statement: Let G be a particular group of a Facebook network with number of people online $(n) \geq 3$, we denote by $deg_1 x$ the number of people with whom x converses (here label edge '+' indicates the flow of information and '-' indicates there is no flow of information)

Then the converse of Ore's theorem in terms of Facebook states that for every pair of people who are not conversing directly i.e., there are not adjacent, the graph has a closed **Hamiltonian labeled path**, then the number of people with whom x converses and the number of people with whom y converses \geq the number of people in a particular group of a face network with **exactly one pair of non adjacent labeled vertices** at a given time. i.e., $deg_1 x + deg_1 y \geq n$.

In the following example we verify the converse.

Example 1

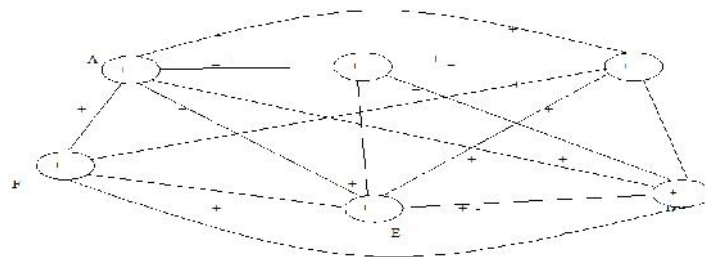
In the below graph person C and person D are not adjacent i.e., there is no information flow between C and D.

The closed Hamiltonian labeled path in the above graph is AFEDBCA

Therefore $deg_1 C + deg_1 D$

$= 3 + 3 = 6 \geq n$ where $n=6$, Therefore $deg_1 C + deg_1 D = n$

Example 1



Therefore we conclude that converse is true in a graph having exactly one pair of non adjacent vertices.

CONCLUSIONS

Every closed Hamiltonian labeled path in Facebook is a 2-way communication between online people. By definition, a Hamiltonian path between two non- adjacent online persons is a one way communication, but Ore's theorem guarantees a closed Hamiltonian path between 2 non-adjacent vertices resulting in 2 way communication, however Ore's theorem is applicable in Facebook only when there is exactly one pair of non-adjacent online persons that is there is no information flow between them at a given instant of time.

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